Derivatives

Topic 7: Differentiation rules (how limits become calculus)

1. Why rules exist (and what they replace)

The derivative is defined by a limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Computing derivatives directly from this definition is correct but inefficient. Differentiation rules are *theorems* that package repeated limit arguments into reusable formulas. The rules do not replace rigor; they compress it.

Pitfall

Do not treat a rule as an identity that holds "because it feels right." Each rule is a theorem with hypotheses (functions must be differentiable where the rule is applied).

2. Linearity (sum rule and constant multiple)

Rule / Theorem

If f and g are differentiable at a, and $c \in \mathbb{R}$, then:

$$(f+g)'(a) = f'(a) + g'(a), \qquad (cf)'(a) = c\,f'(a).$$

Why the rule is true (proof sketch)

Proof idea: write the difference quotient for $f+g$, split the numerator into two parts, then use limit laws:

$$\frac{(f+g)(a+h) - (f+g)(a)}{h} = \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}.$$

Taking $h \to 0$ gives the sum rule.

Worked example

Compute $\frac{d}{dx}(3x^2 - 5x + 7)$.
Using linearity:

$$\frac{d}{dx}(3x^2 - 5x + 7) = 3\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + 0.$$

From Chapter 3 Topic 1, $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(x) = 1$, so

$$= 3(2x) - 5(1) = 6x - 5.$$

3. Product rule

Rule / Theorem

(Product rule.) If f and g are differentiable at a, then fg is differentiable at a and

$$(fg)'(a) = f'(a)\,g(a) + f(a)\,g'(a).$$

Why the rule is true (proof sketch)

Proof (structured and rigorous).
Start with the definition:

$$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Add and subtract $f(a+h)g(a)$ in the numerator (this is the key algebraic move):

$$f(a+h)g(a+h) - f(a)g(a) = \big(f(a+h)g(a+h) - f(a+h)g(a)\big) + \big(f(a+h)g(a) - f(a)g(a)\big).$$

Factor each bracket:

$$= f(a+h)\big(g(a+h) - g(a)\big) + g(a)\big(f(a+h) - f(a)\big).$$

Divide by h:

$$\frac{f(a+h)\big(g(a+h) - g(a)\big)}{h} + g(a)\frac{f(a+h) - f(a)}{h}.$$

Now take limits as $h \to 0$. Since differentiability implies continuity, $f(a+h) \to f(a)$. Also, $\frac{g(a+h)-g(a)}{h} \to g'(a)$ and $\frac{f(a+h)-f(a)}{h} \to f'(a)$. Thus

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Pitfall

The product rule is not $(fg)' = f'g'$. That mistake destroys units in physics and breaks basic examples (try $f(x) = g(x) = x$).

Worked example

Compute $\frac{d}{dx}(x^2 \sin x)$.
Let $f(x) = x^2$, $g(x) = \sin x$. Then $f'(x) = 2x$, $g'(x) = \cos x$. Product rule:

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x.$$

4. Quotient rule

Rule / Theorem

(Quotient rule.) If f and g are differentiable at a and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Why the rule is true (proof sketch)

Proof idea: write $\frac{f}{g} = f \cdot (g^{-1})$ and prove

$$(g^{-1})'(a) = -\frac{g'(a)}{(g(a))^2}$$

using the limit definition, then apply the product rule.

Worked example

Compute $\dfrac{d}{dx}\left(\dfrac{x^2+1}{x-1}\right)$ for $x \neq 1$.

Let $f(x) = x^2 + 1$, $g(x) = x - 1$. Then $f'(x) = 2x$, $g'(x) = 1$. Quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{(2x)(x-1) - (x^2+1)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

Pitfall

If $g(a) = 0$, the formula is not even defined. This matches the reality: $\frac{f}{g}$ is not defined at such points, so differentiability there is impossible.

5. Chain rule (composition)

Definition

(Composition.) If $u = g(x)$ and $y = f(u)$, then the composite function is

$$y = (f \circ g)(x) = f(g(x)).$$

Rule / Theorem

(Chain rule.) If g is differentiable at x and f is differentiable at $u = g(x)$, then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Reminder (term in use)

Mechanism. A small change in x produces a change in $u = g(x)$, which then produces a change in $y = f(u)$. Derivatives multiply because rates of change compose.

Why the rule is true (proof sketch)

Proof idea (rigorous outline). Write

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}.$$

As $h \to 0$, the second factor $\to g'(x)$. If $g(x+h) \neq g(x)$ for small h, the first factor is a difference quotient for f at $u = g(x)$, so it $\to f'(g(x))$. This yields the chain rule. (The full proof handles the case $g(x+h) = g(x)$ by a continuity argument.)

Worked example

Compute $\frac{d}{dx}(\sin(x^3))$.
Let $u = x^3$. Then $\sin(x^3) = \sin(u)$. Chain rule:

$$\frac{d}{dx}\sin(u) = \cos(u) \cdot \frac{du}{dx}.$$

Since $\frac{du}{dx} = 3x^2$, we get

$$\frac{d}{dx}\sin(x^3) = \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

Worked example

Compute $\frac{d}{dx}\sqrt{1 + 2x^2}$.
Write $\sqrt{1 + 2x^2} = (1 + 2x^2)^{1/2}$. Let $u = 1 + 2x^2$. Then $\frac{d}{dx}u = 4x$. Chain rule:

$$\frac{d}{dx}u^{1/2} = \frac{1}{2}u^{-1/2} \cdot u' = \frac{1}{2}(1 + 2x^2)^{-1/2} \cdot 4x = \frac{2x}{\sqrt{1 + 2x^2}}.$$

6. Differentiability domain bookkeeping (where rules apply)

Pitfall

Every derivative formula has an implicit domain condition:

- $\frac{f}{g}$ requires $g(x) \neq 0$.
- $\sqrt{u}$ requires $u(x) \geq 0$ (if working over real numbers).
- $\ln u$ requires $u(x) > 0$.
- $\tan x$ requires $\cos x \neq 0$.

Ignoring domain conditions causes "derivatives" of functions that are not even defined.

7. Exercises (with answers)

Exercises (with answers)

1. Differentiate $f(x) = (x^2 + 1)(x^3 - 2)$.

2. Differentiate $g(x) = \dfrac{\sin x}{x^2 + 1}$.

3. Differentiate $h(x) = \ln(1 + x^2)$.

4. Differentiate $p(x) = (3x - 1)^5$.

Answers.

(1) $f'(x) = 2x(x^3 - 2) + (x^2 + 1) \cdot 3x^2$.

(2) $g'(x) = \dfrac{\cos x\,(x^2 + 1) - \sin x \cdot 2x}{(x^2 + 1)^2}$.

(3) $h'(x) = \dfrac{2x}{1 + x^2}$ (chain rule with $(\ln u)' = u'/u$).

(4) $p'(x) = 5(3x - 1)^4 \cdot 3 = 15(3x - 1)^4$.

Copyright