Derivatives

Topic 6: Derivative as a limit (tangent line + instantaneous rate)

1. The problem: "instantaneous change" is not an algebraic notion

Average change over an interval is algebraic:

$$\text{average rate on } [a, a+h] = \frac{f(a+h) - f(a)}{h}.$$

But "instantaneous rate at a" asks what happens when the interval collapses to a point. There is no algebraic substitute for "collapse"; the only correct mechanism is a limit.

Sanity check / proof idea

Every derivative definition is a limit of average rates. That is why limits are not a preliminary chapter; they are the foundation.

2. Secant slope and tangent slope

Definition

Let f be defined near a. For $h \neq 0$, the **difference quotient** is

$$\frac{f(a+h) - f(a)}{h}.$$

Geometrically, it is the slope of the **secant line** through $(a, f(a))$ and $(a+h, f(a+h))$.

Reminder (term in use)

Secant line. A line through two distinct points of the graph.
Slope. For two points (x_1, y_1), (x_2, y_2), slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

Definition

The **derivative** of f at a (if it exists) is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Reminder (term in use)

Tangent line. The line that best approximates the curve at a point; formally its slope is $f'(a)$ when the derivative exists.
Exists. The limit must exist as a finite real number.

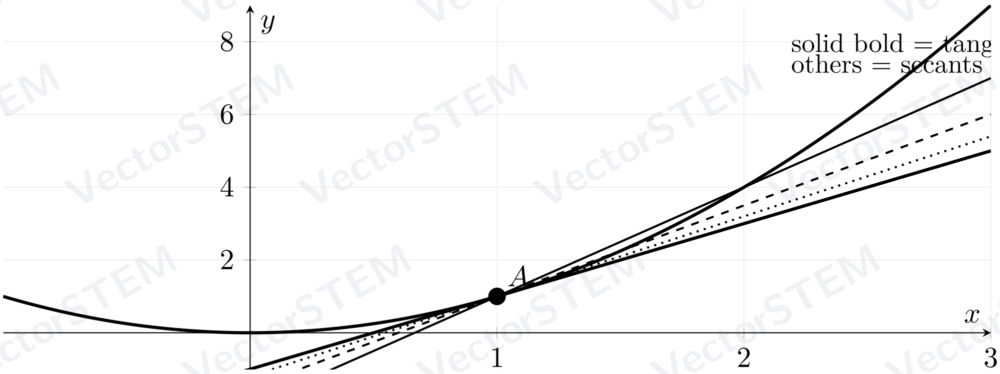
Sanity check / proof idea

Equivalent form (often useful):

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

It is the same definition with $x = a + h$.

3. Picture: secants converging to the tangent



4. First rigorous derivative computation (from the definition)

Worked example

Compute $f'(a)$ for $f(x) = x^2$ using the limit definition.
Start:

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}.$$

Expand:

$$(a+h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2.$$

Factor h:

$$\frac{2ah + h^2}{h} = 2a + h \quad (h \neq 0).$$

Now take the limit:

$$f'(a) = \lim_{h \to 0}(2a + h) = 2a.$$

So $\frac{d}{dx}(x^2) = 2x$.

Sanity check / proof idea

Why this computation is legitimate:

- The cancellation of h is performed for $h \neq 0$, which matches the derivative definition.
- After simplification, a continuous expression $2a + h$ remains, so the limit is immediate.

5. Tangent line equation (how the derivative becomes geometry)

Definition

If $f'(a)$ exists, the **tangent line** at $x = a$ is the line through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Worked example

Find the tangent line to $y = x^2$ at $x = 1$.
Here $f(1) = 1$ and $f'(x) = 2x$, so $f'(1) = 2$. Thus

$$y - 1 = 2(x - 1) \quad \Longrightarrow \quad y = 2x - 1.$$

6. Derivative as instantaneous velocity (physics meaning)

Let $s(t)$ be position at time t. Average velocity on $[t, t + h]$ is

$$v_{\text{avg}} = \frac{s(t + h) - s(t)}{h}.$$

The **instantaneous velocity** is defined by the limit:

$$v(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h} = s'(t).$$

Reminder (term in use)

Instantaneous rate. The derivative is the limit of average rates as the time interval shrinks to zero. This is a definition, not a metaphor.

Worked example

Let $s(t) = t^2$ meters. Then $v(t) = s'(t) = 2t$ m/s. At $t = 3$, the instantaneous velocity is $v(3) = 6$ m/s.

7. Differentiability implies continuity (first structural theorem)

Theorem / Fact

If $f'(a)$ exists, then f is continuous at a.

Sanity check / proof idea

Proof idea (clean and rigorous).
Assume $f'(a)$ exists. Then the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

exists and is finite. Multiply by h:

$$f(a+h)-f(a) = h \cdot \frac{f(a+h)-f(a)}{h}.$$

As $h \to 0$, the factor $h \to 0$, while the quotient approaches $f'(a)$ (finite). Hence the product tends to 0, so

$$\lim_{h\to 0} (f(a+h)-f(a)) = 0 \quad \Rightarrow \quad \lim_{h\to 0} f(a+h) = f(a).$$

That is $\lim_{x\to a} f(x) = f(a)$: continuity at a.

Pitfall

Continuity does not imply differentiability. Example: $f(x) = |x|$ is continuous at 0 but not differentiable there (left slope -1, right slope $+1$).

8. Exercises (with answers)

Exercises (with answers)

1. Using the definition, compute the derivative of $f(x) = x^3$. (Hint: expand $(a+h)^3$.)

2. Find the tangent line to $y = x^2$ at $x = 2$.

3. Let $f(x) = |x|$. Compute the difference quotient at $a = 0$ and show the derivative at 0 does not exist.

4. Prove: if $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$. (Write a full proof.)

Answers.
(1) $f'(a) = 3a^2$.
(2) $f(2) = 4$, $f'(2) = 4 \Rightarrow y - 4 = 4(x-2) \Rightarrow y = 4x - 4$.
(3) For $h \neq 0$, $\frac{|h|-0}{h} = \frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$. Limits disagree.
(4) Same proof as in the theorem box.

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