

Limits and Continuity

Topic 5: Core theorems of continuity (IVT, EVT, boundedness)

1. Why continuity theorems matter

Continuity is not only a definition. Its power is that it implies *global* consequences on intervals: existence of roots, attainment of maxima/minima, boundedness, and more. These results are the first place where calculus begins to resemble higher analysis: a local condition (continuity) forces nontrivial global structure.

Sanity check / proof idea

The phrase “continuous on a closed interval” is a stability assumption. Without it, IVT and EVT can fail in dramatic ways.

2. Intermediate Value Theorem (IVT)

Theorem

(Intermediate Value Theorem.) If f is continuous on $[a, b]$ and N is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that

$$f(c) = N.$$

Reminder (term in use)

Between. N is between $f(a)$ and $f(b)$ means either $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$.
Existence only. IVT guarantees at least one c , not uniqueness.

Worked example

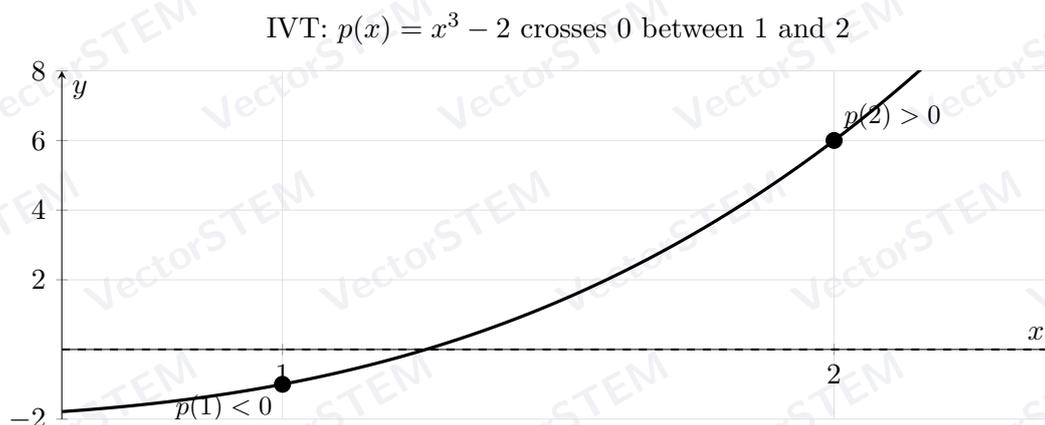
(Existence of a root without solving.) Let $p(x) = x^3 - 2$. On $[1, 2]$,

$$p(1) = -1, \quad p(2) = 6.$$

Since 0 lies between -1 and 6 , IVT implies there exists $c \in (1, 2)$ such that

$$p(c) = 0 \iff c^3 = 2.$$

Hence $\sqrt[3]{2} \in (1, 2)$ exists as a real number (and is the root of $x^3 - 2$).

**Pitfall**

What IVT does not say. It does not give a formula for c . It does not say how many such c exist. It asserts existence forced by continuity.

3. Boundedness Theorem (BVT)**Theorem**

(Boundedness Theorem.) If f is continuous on a closed interval $[a, b]$, then f is bounded on $[a, b]$. That is, there exists $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all } x \in [a, b].$$

Reminder (term in use)

Bounded. A function is bounded on a set if its absolute value is bounded above by some constant on that set.

Closed interval. Includes endpoints; $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Sanity check / proof idea

Why “closed interval” matters: continuity alone on an open interval (a, b) does not guarantee boundedness. Example: $f(x) = \frac{1}{x-a}$ is continuous on (a, b) but unbounded near a .

Worked example

On $[0, 1]$, $f(x) = \sqrt{x}$ is continuous and bounded:

$$0 \leq \sqrt{x} \leq 1.$$

This is not deep by itself; the theorem is deep because it guarantees boundedness for *any* continuous function on $[a, b]$, without an explicit formula for the bound.

4. Extreme Value Theorem (EVT)

Theorem

(Extreme Value Theorem.) If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$. That is, there exist points $x_{\max}, x_{\min} \in [a, b]$ such that

$$f(x_{\max}) \geq f(x) \quad \text{and} \quad f(x_{\min}) \leq f(x) \quad \text{for all } x \in [a, b].$$

Reminder (term in use)

Attains. The maximum/minimum is achieved at an actual point in the interval (not just approached).

Absolute maximum/minimum. Global extrema on the entire interval, not local.

Worked example

Let $f(x) = x(1-x)$ on $[0, 1]$. It is continuous on $[0, 1]$, so EVT guarantees both a maximum and a minimum.

Compute explicitly:

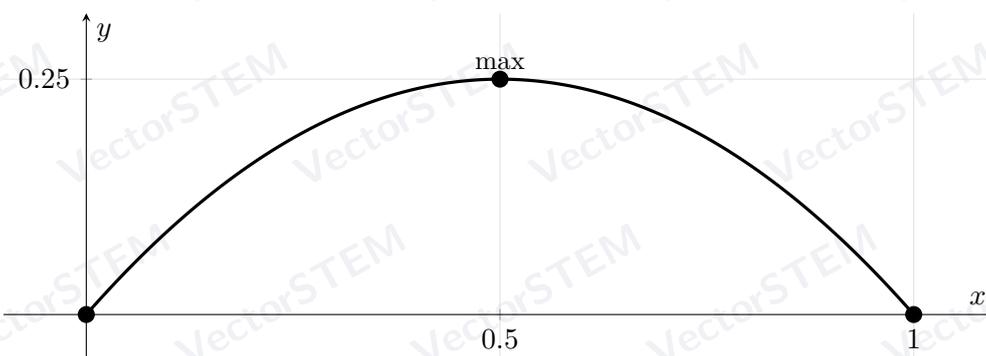
$$f(x) = x - x^2, \quad f'(x) = 1 - 2x.$$

Critical point: $f'(x) = 0 \Rightarrow x = \frac{1}{2}$. Evaluate:

$$f(0) = 0, \quad f(1) = 0, \quad f\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Thus the absolute maximum is $\frac{1}{4}$ at $x = \frac{1}{2}$, and the absolute minimum is 0 at both endpoints.

EVT: $f(x) = x(1-x)$ attains max on $[0, 1]$



Pitfall

Why EVT can fail without closedness. On $(0, 1)$, the function $f(x) = x$ is continuous but has no maximum (it can get arbitrarily close to 1 but never equals 1 within the interval).

5. Bolzano's Theorem (root existence as a special case of IVT)

Theorem

(Bolzano's Theorem.) If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Sanity check / proof idea

Reason: $f(a)f(b) < 0$ means $f(a)$ and $f(b)$ have opposite signs, so 0 lies between them. Apply IVT with $N = 0$.

6. Exercises (with answers)

Exercises (with answers)

1. Use IVT to show that $x^3 + x - 1 = 0$ has a real root. Provide an interval.
2. Give an example of a function continuous on $(0, 1)$ that is unbounded on $(0, 1)$.
3. On $[0, 2]$, find absolute extrema of $f(x) = x^2 - 2x$.
4. Explain in one sentence why EVT requires a closed interval.

Answers.

- (1) $f(0) = -1$, $f(1) = 1 \Rightarrow$ root in $(0, 1)$.
- (2) $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ and unbounded near 0.
- (3) $f'(x) = 2x - 2 = 0 \Rightarrow x = 1$. Values: $f(0) = 0$, $f(1) = -1$, $f(2) = 0$. Max = 0 at 0, 2; min = -1 at 1.
- (4) Because a continuous function can approach but not attain an extreme on an open interval (e.g. $f(x) = x$ on $(0, 1)$).

Copyright

© 2026 VectorSTEM. All rights reserved.