

# Limits and Continuity

Topic 4: Limits at infinity, infinite limits, and asymptotes

## 1. Two different “infinity” situations

There are two distinct ideas:

- **Limits at infinity:**  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ . This describes behavior far to the right/left.
- **Infinite limits:**  $f(x) \rightarrow \infty$  or  $f(x) \rightarrow -\infty$  as  $x \rightarrow a$ . This describes blow-up near a finite point.

### Pitfall

Infinity is not a number. Expressions like “plug in  $x = \infty$ ” are not part of the definition. All statements are defined using  $\varepsilon$ -style inequalities and quantifiers.

## 2. Limits at infinity (precise definition)

### Definition

(Definition of  $\lim_{x \rightarrow \infty} f(x) = L$ .) The statement

$$\lim_{x \rightarrow \infty} f(x) = L$$

means: for every  $\varepsilon > 0$  there exists a number  $M$  such that

$$x > M \implies |f(x) - L| < \varepsilon.$$

### Definition

(Definition of  $\lim_{x \rightarrow -\infty} f(x) = L$ .) The statement

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means: for every  $\varepsilon > 0$  there exists a number  $M$  such that

$$x < M \implies |f(x) - L| < \varepsilon.$$

### Reminder (term in use)

$M$ . A threshold beyond which  $x$  is “sufficiently large” (or “sufficiently negative”).

**Key point.** For  $x$  far enough out,  $f(x)$  stays inside the  $\varepsilon$ -band around  $L$ .

### 3. Horizontal asymptotes

#### Definition

The line  $y = L$  is a **horizontal asymptote** of  $f$  if

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L.$$

#### Worked example

Compute  $\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 - 5}$ .

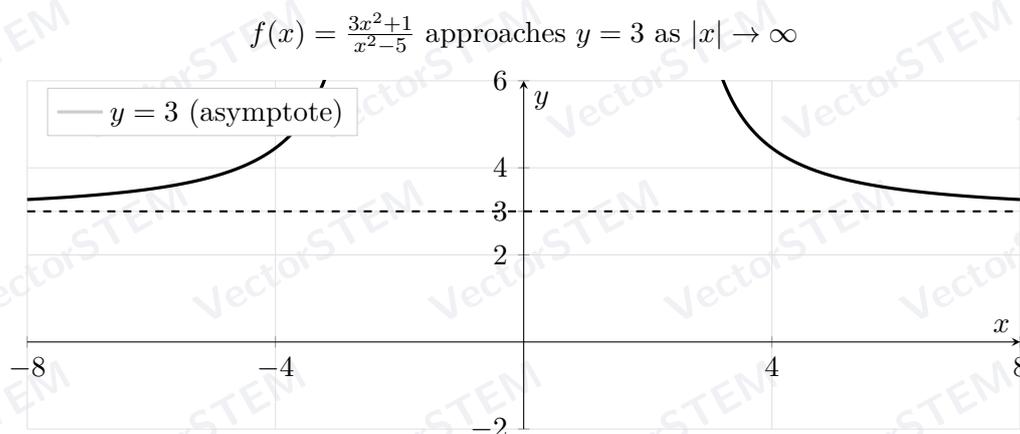
Divide numerator and denominator by  $x^2$ :

$$\frac{3x^2 + 1}{x^2 - 5} = \frac{3 + \frac{1}{x^2}}{1 - \frac{5}{x^2}}.$$

As  $x \rightarrow \infty$ ,  $\frac{1}{x^2} \rightarrow 0$  and  $\frac{5}{x^2} \rightarrow 0$ , so by limit laws:

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 - 5} = \frac{3 + 0}{1 - 0} = 3.$$

Hence  $y = 3$  is a horizontal asymptote.



### 4. Infinite limits (precise definition)

#### Definition

(Definition of  $\lim_{x \rightarrow a} f(x) = \infty$ .) The statement

$$\lim_{x \rightarrow a} f(x) = \infty$$

means: for every  $N > 0$  there exists  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies f(x) > N.$$

**Definition**

(Definition of  $\lim_{x \rightarrow a} f(x) = -\infty$ .) The statement

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means: for every  $N > 0$  there exists  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies f(x) < -N.$$

**Reminder (term in use)**

**Why  $N$  replaces  $\varepsilon$ .** The goal is not “close to a finite  $L$ ” but “exceeds any chosen bound.”  
**Same structure.** A user chooses a target bound  $N$ ; a proof supplies  $\delta$  that forces  $f(x)$  past that bound near  $a$ .

**5. Vertical asymptotes****Definition**

The line  $x = a$  is a **vertical asymptote** of  $f$  if at least one of the following holds:

$$\lim_{x \rightarrow a^+} f(x) = \infty, \quad \lim_{x \rightarrow a^+} f(x) = -\infty, \quad \lim_{x \rightarrow a^-} f(x) = \infty, \quad \lim_{x \rightarrow a^-} f(x) = -\infty.$$

**Worked example**

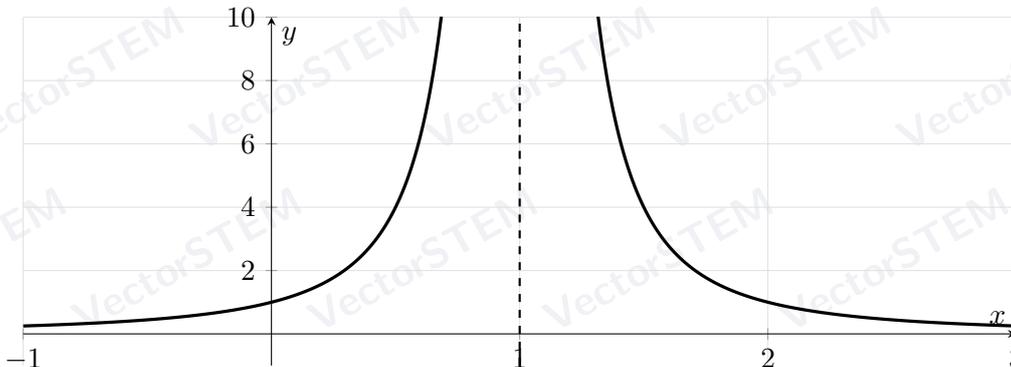
For  $f(x) = \frac{1}{(x-1)^2}$ , show that  $x = 1$  is a vertical asymptote.

Given any  $N > 0$ , choose  $\delta = \frac{1}{\sqrt{N}}$ . If  $0 < |x - 1| < \delta$ , then  $|x - 1| < \frac{1}{\sqrt{N}}$ , so

$$(x - 1)^2 < \frac{1}{N} \implies \frac{1}{(x - 1)^2} > N.$$

Thus  $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$ , so  $x = 1$  is a vertical asymptote.

$f(x) = \frac{1}{(x-1)^2}$  and the vertical asymptote  $x = 1$



## 6. A computation toolkit (standard patterns)

### Rational functions at infinity (degree rule)

Let  $p(x)$  and  $q(x)$  be polynomials and consider  $\frac{p(x)}{q(x)}$ . Let  $m = \deg p$ ,  $n = \deg q$ .

- If  $m < n$ , then  $\lim_{x \rightarrow \pm\infty} \frac{p(x)}{q(x)} = 0$ .
- If  $m = n$ , then the limit is the ratio of leading coefficients.
- If  $m > n$ , the function does not have a finite horizontal asymptote; it may have an oblique (slant) asymptote.

### Reminder (term in use)

**Leading coefficient.** The coefficient of the highest-power term. Example: for  $3x^2 + 1$ , it is 3.

**Degree.** The highest exponent with nonzero coefficient.

## 7. Exercises (with answers)

### Exercises (with answers)

1. Compute  $\lim_{x \rightarrow \infty} \frac{5x^3 - 2}{2x^3 + 7x}$ .
2. Compute  $\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 1}$ .
3. Show  $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$  using the precise definition.
4. Determine the horizontal asymptote(s) of  $f(x) = \frac{2x^2 - 1}{x^2 + 4}$ .

### Answers.

- (1) Degrees equal  $\Rightarrow$  ratio of leading coefficients =  $5/2$ .
- (2) Divide by  $x^2 \Rightarrow \frac{1}{1+1/x^2} \rightarrow 1$ .
- (3) Given  $N > 0$ , choose  $\delta = \frac{1}{N}$ . If  $0 < x < \delta$ , then  $\frac{1}{x} > N$ .
- (4) Degrees equal  $\Rightarrow$  asymptote  $y = 2$  (both sides).

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