Limits and Continuity

Topic 3: Precise definition of limit and the limit laws

1. Precise definition of $\lim_{x\to a} f(x) = L$

Definition

(ε–δ definition.) Let f be defined on some punctured neighborhood of a (i.e., for x near a, possibly excluding a). Then

$$\lim_{x\to a} f(x) = L$$

means:

for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Reminder (term in use)

ε. Output tolerance: how close $f(x)$ must be to L.
δ. Input tolerance: how close x must be to a.
Punctured neighborhood. The condition $0 < |x - a| < \delta$ means "x is within δ of a, but $x \neq a$".

Sanity check / proof idea

How to read the definition as a contract:

- A user picks $\varepsilon > 0$ (the demanded output accuracy).
- A proof must respond with a $\delta > 0$ (an input radius) that forces the output accuracy.
- The implication must hold for *all* x satisfying $0 < |x - a| < \delta$.

This is why the statement "the limit is L" is strong: it guarantees control of $f(x)$ near a at any requested precision.

2. Uniqueness of limits (why a limit cannot have two values)

Theorem / Limit Law

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

Sanity check / proof idea

Proof idea (fully rigorous in one line): If $L \neq M$, choose $\varepsilon = \frac{|L-M|}{3}$. The definition forces $f(x)$ to be within ε of both L and M for x sufficiently close to a, which is impossible because the two ε-intervals around L and M are disjoint.

3. Limit laws (statements)

Reminder (term in use)

Limit law format. Each law has the form: if the limits on the right exist, then the limit on the left exists and equals the stated expression. These are *theorems* proved from the ε–δ definition.

Limit laws (algebraic)

Assume $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$. Then:

1. $\lim\limits_{x\to a} (f(x) + g(x)) = A + B$.
2. $\lim\limits_{x\to a} (f(x) - g(x)) = A - B$.
3. $\lim\limits_{x\to a} (c\, f(x)) = cA$ for any constant $c \in \mathbb{R}$.
4. $\lim\limits_{x\to a} (f(x)\, g(x)) = AB$.
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$, provided $B \neq 0$.
6. $\lim\limits_{x\to a} (f(x))^n = A^n$ for integers $n \geq 1$.

Limit laws (order and squeeze)

Assume $f(x) \leq g(x) \leq h(x)$ for all x near a (excluding a if needed).

1. (**Order preservation**) If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, and $f(x) \leq g(x)$ near a, then $A \leq B$.
2. (**Squeeze theorem**) If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Pitfall

Limits are not always computed by substitution. Substitution works only when the function is continuous at the point (a theorem proved later). Indeterminate forms like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ mean algebraic transformation is required.

4. Why the sum law is true (a model proof)

Theorem (sum law)

If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

$$\lim_{x\to a} (f(x) + g(x)) = A + B.$$

Sanity check / proof idea

Proof (epsilon–delta, complete and readable).
Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$0 < |x - a| < \delta \;\Rightarrow\; |(f(x) + g(x)) - (A + B)| < \varepsilon.$$

Start with the algebra:

$$|(f(x) + g(x)) - (A + B)| = |(f(x) - A) + (g(x) - B)|.$$

Use the triangle inequality (reminder: $|u + v| \le |u| + |v|$):

$$|(f(x) - A) + (g(x) - B)| \le |f(x) - A| + |g(x) - B|.$$

Now force each term to be $< \varepsilon/2$. Because $\lim_{x\to a} f(x) = A$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2$. Because $\lim_{x\to a} g(x) = B$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon/2$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta$ implies both inequalities hold, hence

$$|(f(x) + g(x)) - (A + B)| \le |f(x) - A| + |g(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the sum law.

Reminder (term in use)

Triangle inequality (reminder). For all real numbers u, v,

$$|u + v| \le |u| + |v|.$$

It is the basic tool that turns “control each piece” into “control the sum”.

5. Worked computations using limit laws

Worked example

Compute $\lim_{x\to 3}(2x^2 - 5x + 7)$.
Polynomials are built from x using addition and multiplication, and limit laws allow term-by-term evaluation:

$$\lim_{x\to 3}(2x^2 - 5x + 7) = 2 \cdot \big(\lim_{x\to 3} x\big)^2 - 5 \cdot \lim_{x\to 3} x + 7.$$

Since $\lim_{x\to 3} x = 3$, this becomes

$$2 \cdot 3^2 - 5 \cdot 3 + 7 = 18 - 15 + 7 = 10.$$

Worked example

Compute $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.
Direct substitution gives $\frac{0}{0}$, an indeterminate form. Algebraic simplification is required:

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2 \quad (x \neq 2).$$

Therefore

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

6. Squeeze theorem: the mechanism, not just the slogan

Worked example

Show $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.
Use the bound $-1 \leq \sin(\cdot) \leq 1$. Multiply by $x^2 \geq 0$:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Now compute the outer limits:

$$\lim_{x\to 0}(-x^2) = 0, \qquad \lim_{x\to 0} x^2 = 0.$$

By the squeeze theorem, the middle function has limit 0.

Reminder (term in use)

Squeeze theorem (reminder). If $f \leq g \leq h$ near a and $\lim f = \lim h = L$, then $\lim g = L$. The point is: $g(x)$ is trapped in a shrinking corridor.

7. Exercises (with answers)

Exercises (with answers)

1. Using the ε–δ definition, prove that $\lim_{x\to a}(x) = a$.

2. Prove the constant multiple law: if $\lim_{x\to a} f(x) = A$, then $\lim_{x\to a}(cf(x)) = cA$.

3. Compute $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$.

4. Show $\lim\limits_{x\to 0} x\sin(\frac{1}{x}) = 0$.

Answers.
(1) Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $0 < |x-a| < \delta \Rightarrow |x-a| < \varepsilon$.
(2) Use $|cf(x) - cA| = |c| \cdot |f(x) - A|$ and choose δ for $\varepsilon/|c|$ (if $c \neq 0$).
(3) Factor $x^3 - 1 = (x-1)(x^2+x+1) \Rightarrow$ limit $= 1^2 + 1 + 1 = 3$.
(4) Use $|\sin(\cdot)| \leq 1 \Rightarrow |x\sin(1/x)| \leq |x|$ and squeeze to 0.

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