Limits and Continuity

Topic 2: Continuity — definition, mechanisms, and standard theorems

1. What continuity is actually asserting

Continuity is a statement that connects three things at a point a:

(i) the value $f(a)$, (ii) nearby values $f(x)$, (iii) the limit as $x \to a$.

Intuitively: if x is close to a, then $f(x)$ is close to $f(a)$. Rigorously: this is exactly the limit definition with $L = f(a)$.

Sanity check (why it works)

Continuity is not a visual slogan ("no holes"). It is a precise implication:

$$x \to a \quad \Rightarrow \quad f(x) \to f(a).$$

2. Continuity at a point

Definition

(Continuity at a.) Let f be defined at a. The function f is **continuous at** a if

$$\lim_{x \to a} f(x) = f(a).$$

Reminder (term in use)

Defined at a. The value $f(a)$ exists (as a real number).
Continuous at a. The limit exists and equals the function value.

Sanity check (why it works)

The definition can be split into three checkable conditions:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$.

Failure of continuity occurs when at least one condition fails.

3. ε–δ form of continuity

Definition

(ε–δ definition of continuity.) A function f is continuous at a if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon.$$

Reminder (term in use)

Compare with the limit definition:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

For continuity, $L = f(a)$ and the condition $x \neq a$ is unnecessary because $f(a)$ is already defined.

4. Standard discontinuities (classified by what fails)

Definition

Discontinuity at a occurs when at least one of the three conditions fails:

- **(A)** $f(a)$ does not exist,
- **(B)** $\lim_{x\to a} f(x)$ does not exist,
- **(C)** the limit exists but is not equal to $f(a)$.

4.1 Removable discontinuity (limit exists, value mismatched or missing)

Worked example

Define

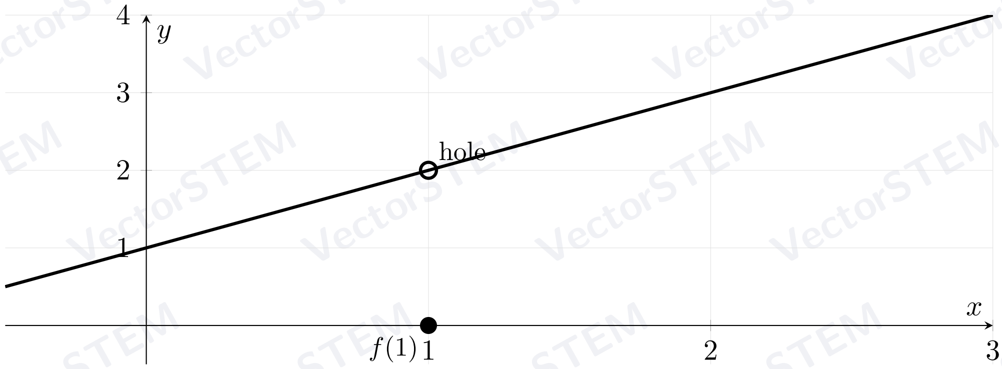
$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

For $x \neq 1$, $f(x) = x + 1$. Therefore

$$\lim_{x\to 1} f(x) = 2, \qquad \text{but} \qquad f(1) = 0.$$

So continuity fails by condition (C). The discontinuity is *removable*: redefining $f(1) = 2$ makes the function continuous at 1.

Removable discontinuity at $x = 1$



4.2 Jump discontinuity

Worked example

Let

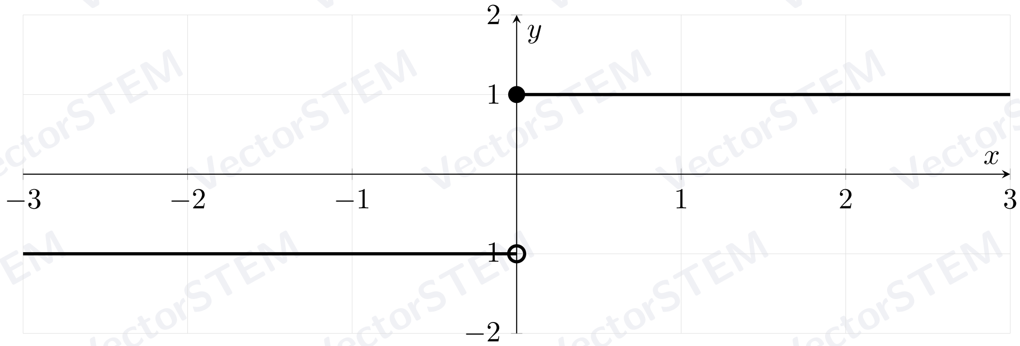
$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then

$$\lim_{x \to 0^-} g(x) = -1, \qquad \lim_{x \to 0^+} g(x) = 1.$$

The one-sided limits differ, so $\lim_{x \to 0} g(x)$ does not exist. Continuity fails by condition (B).

Jump discontinuity at $x = 0$

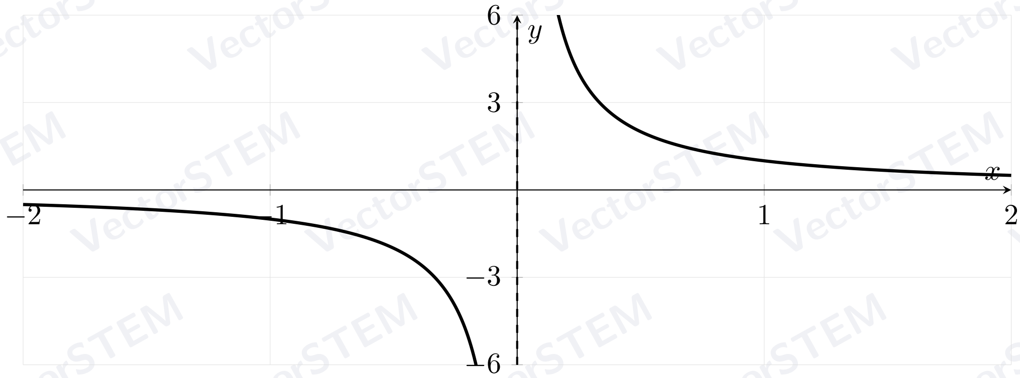


4.3 Infinite discontinuity

Worked example

Consider $h(x) = \dfrac{1}{x}$ near $x = 0$. As $x \to 0^+$, $h(x) \to +\infty$, and as $x \to 0^-$, $h(x) \to -\infty$. There is no finite real number L such that $h(x) \to L$ as $x \to 0$. Thus continuity fails by (A) and (B) at 0 (the function is not defined at 0, and no finite limit exists).

Infinite discontinuity: $y = \frac{1}{x}$



5. Continuity on an interval; left/right continuity

Definition

A function is **continuous on an interval** if it is continuous at every point of that interval. At an endpoint a of an interval $[a, b]$, continuity is interpreted one-sided:

$$\text{continuous at } a \text{ on } [a,b] \quad \Longleftrightarrow \quad \lim_{x\to a^+} f(x) = f(a).$$

Similarly at b: $\lim_{x\to b^-} f(x) = f(b)$.

Reminder (term in use)

Endpoint. A boundary point of the interval (e.g. a in $[a, b]$).
One-sided limit. A limit taken with x restricted to one side of the point.

6. Continuity is stable under algebraic operations

Definition

If f and g are continuous at a, then the following are continuous at a:

$$f + g, \quad f - g, \quad fg, \quad \frac{f}{g} \text{ (provided } g(a) \neq 0).$$

Also, if f is continuous at a and g is continuous at $f(a)$, then the composition $g \circ f$ is continuous at a.

Reminder (term in use)

Composition. $(g \circ f)(x) = g(f(x))$. The output of f becomes the input of g.

Sanity check (why it works)

This is the reason polynomials and rational functions are continuous on their domains: polynomials are built from constants and x using addition and multiplication; rational functions are quotients of polynomials, so continuity holds wherever the denominator is nonzero.

7. The Intermediate Value Theorem

Definition

(Intermediate Value Theorem, IVT.) If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that

$$f(c) = N.$$

Reminder (term in use)

Between. N is between $f(a)$ and $f(b)$ means either $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$.
Existence. IVT guarantees at least one such c, not uniqueness.

Worked example

Let $f(x) = x^3 - 2$. On $[1, 2]$:

$$f(1) = -1, \qquad f(2) = 6.$$

The value 0 lies between -1 and 6. Since polynomials are continuous, IVT guarantees some $c \in [1, 2]$ with

$$c^3 - 2 = 0 \quad \Rightarrow \quad c = \sqrt[3]{2}.$$

IVT gives existence of a root in $[1, 2]$ without solving the equation explicitly.

8. Exercises (with answers)

Exercises (with answers)

1. Check continuity at $x = 1$ for

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

If not continuous, give a value that makes it continuous.

2. Let $g(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$. Compute the one-sided limits at 0 and decide whether g is continuous at 0.

3. Prove from the ε–δ definition that the function $f(x) = 3x - 5$ is continuous at every $a \in \mathbb{R}$. (Hint: $|f(x) - f(a)| = 3|x - a|$.)

4. Use IVT to justify that $x^5 + x - 1 = 0$ has a real root. (Choose a simple interval.)

Answers.
(1) $\lim_{x\to 1} f(x) = 2 \neq f(1)$; set $f(1) = 2$.
(2) Left $= -1$, right $= 1$; not continuous at 0.
(3) Given $\varepsilon > 0$, choose $\delta = \varepsilon/3$. Then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.
(4) Let $p(x) = x^5 + x - 1$. Then $p(0) = -1$, $p(1) = 1$. By IVT there is $c \in (0, 1)$ with $p(c) = 0$.

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