Limits and Continuity

Topic 1: Limits — meaning, computation, and geometry

1. Why limits exist (the problem that forces the definition)

Many expressions in calculus describe behavior *near* a point, not necessarily *at* the point. For example, the slope of a tangent line involves slopes of secant lines that approach a limit as the two points merge. To formalize "approach" without ambiguity, the concept of limit is introduced.

Sanity check (why it works)

A limit is not "plugging in." A limit describes what $f(x)$ is forced to do when x is forced close to a point. The value $f(a)$ may be irrelevant or even undefined.

2. Informal picture (what the notation claims)

Definition

The statement

$$\lim_{x\to a} f(x) = L$$

means: by choosing x sufficiently close to a (but not necessarily equal to a), the value $f(x)$ becomes as close as desired to L.

Reminder (term in use)

"$x \to a$" (approach). Inputs x are taken near a.
Limit value L. The number that $f(x)$ is forced close to when x is forced close to a.
Important: the limit concerns values of $f(x)$ for $x \neq a$.

3. The ε–δ definition (complete rigor)

Definition

(ε–δ definition.) Let f be defined on some punctured neighborhood of a (meaning: on values near a, possibly excluding a). Then

$$\lim_{x\to a} f(x) = L$$

means:

for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Reminder (term in use)

ε **(epsilon).** A chosen tolerance for output error: "how close to L".
δ **(delta).** A required tolerance for input distance: "how close to a".
Punctured neighborhood. Values of x near a with $x \neq a$; this is exactly what $0 < |x-a|$ encodes.

Sanity check (why it works)

The logic is one-directional:

- A user chooses $\varepsilon > 0$ (demanding output accuracy).
- A proof must produce some $\delta > 0$ (an input radius) that guarantees the demanded accuracy.

The definition turns the informal claim "$f(x)$ gets close to L" into a checkable statement.

4. Visual geometry of ε and δ

Geometric meaning: keeping $f(x)$ inside an ε-band by restricting x to a δ-window

5. Two fundamental facts that prevent common confusion

Definition

If $\lim_{x\to a} f(x) = L$, then the values of $f(x)$ near a are forced near L. The definition does *not* require $f(a)$ to exist, and does *not* require $f(a) = L$.

Pitfall

Common false statement. "If the limit exists, then $f(a)$ must equal the limit."
Reality. The limit concerns $x \neq a$. The value at a can be different, or undefined.

Worked example

Define

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

For $x \neq 1$,

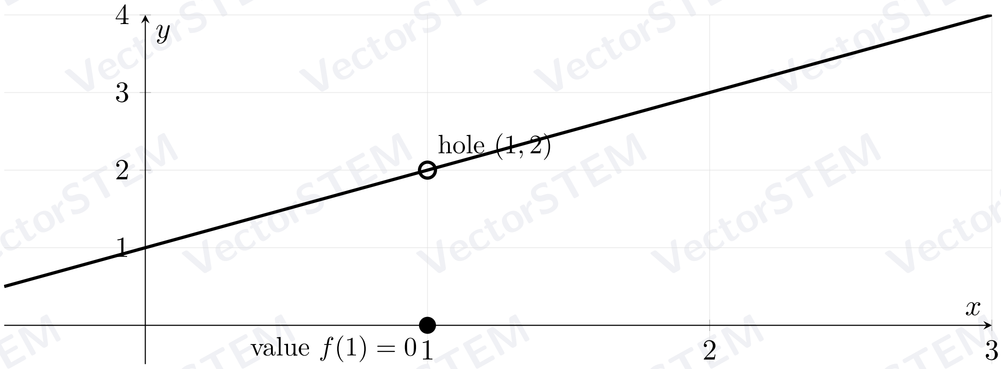
$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1.$$

Hence

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2,$$

but $f(1) = 0$. The limit exists and differs from the function value at the point.

Removable discontinuity: limit exists but $f(1) \neq \lim_{x\to 1} f(x)$



6. One-sided limits and why they matter

Definition

The **left-hand limit** and **right-hand limit** at a are:

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L,$$

meaning the limit is taken with $x < a$ or $x > a$, respectively.
The two-sided limit $\lim_{x\to a} f(x)$ exists and equals L if and only if both one-sided limits exist and are equal to L.

Reminder (term in use)

$x \to a^-$. Approach a from the left (values smaller than a).
$x \to a^+$. Approach a from the right (values larger than a).

Worked example

Let

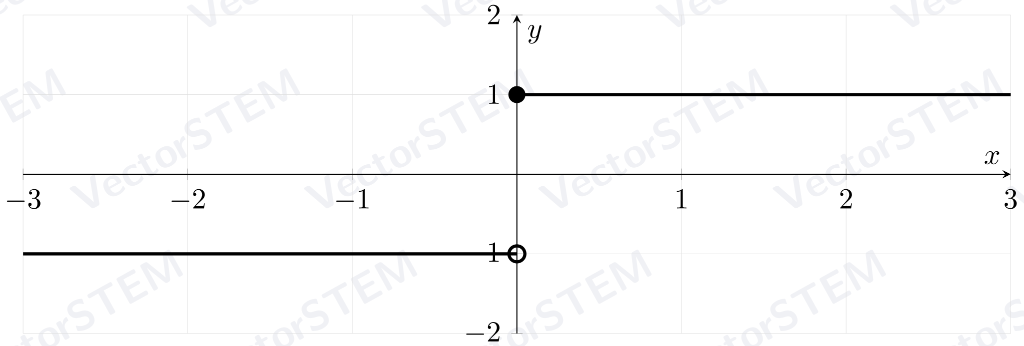
$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then

$$\lim_{x \to 0^-} f(x) = -1, \qquad \lim_{x \to 0^+} f(x) = 1,$$

so $\lim_{x \to 0} f(x)$ does not exist (the one-sided limits disagree).

Jump: one-sided limits differ at 0



7. How to compute limits (without sacrificing correctness)

Definition

When standard algebraic rules apply, limits can be computed by:

- algebraic simplification (factor/cancel, rationalize),
- known limits + limit laws,
- squeeze theorem (when trapped between two functions with same limit),
- one-sided analysis for piecewise functions.

Reminder (term in use)

Limit laws (reminder). If $\lim f = A$ and $\lim g = B$, then (when defined):

$$\lim(f+g) = A + B, \quad \lim(fg) = AB, \quad \lim \frac{f}{g} = \frac{A}{B} \ (B \neq 0).$$

These laws are theorems (provable from ε–δ definition).

Worked example

Compute $\lim_{x\to 2} \frac{x^2-4}{x-2}$.
Factor:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \quad (x \neq 2).$$

Then

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Pitfall

Danger signal. A form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indicates that direct substitution is not valid. It does not mean "the limit is 0" or "the limit is undefined"; it means the expression must be transformed.

8. Exercises (with answers)

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1. Let $f(x) = 3x - 5$. Compute $\lim_{x\to 2} f(x)$.

2. Compute $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

3. For $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$, compute $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, and decide whether $\lim_{x\to 0} f(x)$ exists.

4. Prove using the ε–δ definition that $\lim_{x\to a}(x) = a$. (Hint: choose $\delta = \varepsilon$.)

Answers.
(1) $3 \cdot 2 - 5 = 1$.
(2) $x^2 - 1 = (x-1)(x+1) \Rightarrow$ limit $= 2$.
(3) Left $= -1$, right $= 1$, so two-sided limit does not exist.
(4) If $0 < |x-a| < \delta$ then $|x-a| < \varepsilon$ by taking $\delta = \varepsilon$; hence the limit equals a.

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