The Cello as a Complex Wave System

Part 1: The Physics of the Bow — Stick-Slip and the Wave Equation

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About this series. This is Part 1 of a four-part research program treating the cello as a controllable prototype of a complex wave system. Each part is self-contained. The series culminates in a full research article with computational results and real acoustic measurements.

1. Why the Cello?

Consider three physical systems: a cello string under a moving bow, a plasma column carrying an oscillating current, and a seismic fault under tectonic stress. At first glance they share nothing. But they are described by the same mathematical object — a wave field driven by a nonlinear boundary condition — and they all exhibit the same qualitative behaviour: long periods of coherent oscillation interrupted by sudden, noisy transitions.

The cello is the most accessible of these. It can be excited reproducibly, recorded with a microphone, and its control parameter (bow force) varied continuously by hand. This makes it an ideal *prototype* for developing analysis tools transferable to less accessible wave systems.

This paper derives the physics from first principles. No prior knowledge of music acoustics is assumed; only classical mechanics and Fourier analysis.

2. The Free String: Wave Equation and Normal Modes

2.1. Derivation of the Wave Equation

Consider a flexible string of length L, linear mass density μ (kg/m), and tension T (N). We assume small transverse displacements $u(x, t)$. For a small element of length dx at position x, Newton's second law in the transverse direction gives:

$$\begin{aligned}\mu\, dx \cdot \frac{\partial^2 u}{\partial t^2} &= T \sin\theta(x + dx, t) - T \sin\theta(x, t) \\ &\approx T \frac{\partial \theta}{\partial x}\, dx = T \frac{\partial^2 u}{\partial x^2}\, dx,\end{aligned} \tag{1}$$

where we used the small-angle approximation $\sin\theta \approx \partial u/\partial x$. Dividing by $\mu\, dx$:

The 1D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c = \sqrt{\frac{T}{\mu}}. \tag{2}$$

c is the phase speed of transverse waves. For the cello C2 string: $T \approx 40\,\mathrm{N}$, $\mu \approx 2.4\,\mathrm{g/m}$, giving $c \approx 129\,\mathrm{m/s}$.

2.2. Normal Modes and Eigenfrequencies

For a string clamped at both ends ($u(0,t) = u(L,t) = 0$), separable solutions $u = X(x)T(t)$ give:

$$X(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

Normal Mode Frequencies

$$f_n = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \qquad n = 1, 2, 3, \ldots \tag{3}$$

The fundamental $f_1 = c/(2L)$ sets the pitch. All $f_n = nf_1$ are exact integer multiples — the **harmonic series**.

Numbers for the cello C2 string ($f_1 = 65.4\,\mathrm{Hz}$, $L \approx 0.69\,\mathrm{m}$): $f_2 = 130.8\,\mathrm{Hz}$, $f_3 = 196.2\,\mathrm{Hz}$, $f_4 = 261.6\,\mathrm{Hz}$,...
The violin G3 string has $f_1 = 196\,\mathrm{Hz}$ — the same as the cello's *third* harmonic. This overlap explains why the instruments sound related but distinct.

2.3. General Solution and Energy

The general solution is a superposition of all normal modes:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos(2\pi f_n t) + b_n \sin(2\pi f_n t)\right]. \tag{4}$$

The energy in mode n is $E_n = \frac{1}{4}\mu L \omega_n^2 (a_n^2 + b_n^2)$. The spectral energy distribution $\{E_n\}$ is the fundamental quantity we estimate from the acoustic signal via FFT.

3. The Driven String: Bow Force as a Nonlinear Boundary Condition

3.1. Modified Wave Equation

The bow exerts a transverse force concentrated at contact point $x = x_b$:

Driven Wave Equation

$$\mu\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} + F_b(t)\,\delta(x - x_b), \tag{5}$$

where $\delta(x - x_b)$ is the Dirac delta and $F_b(t)$ is the bow force.

The bow force is not a given function of time — it depends on the motion of the string itself, making (5) a **nonlinear integro-differential equation**.

3.2. The Friction Force

Define the **relative velocity**: $v_{\text{rel}}(t) = v_b - \dot{u}(x_b, t)$, where v_b is the bow velocity. The Woodhouse (1993) friction model gives:

Nonlinear Friction Model

$$F_b(t) = F_N \cdot \mu_{\text{eff}}(v_{\text{rel}}), \tag{6}$$

$$\mu_{\text{eff}}(v) = \begin{cases} \mu_s & v = 0 \text{ (stick)}, \\ \dfrac{\mu_k}{1 + |v|/v_0} & v \neq 0 \text{ (slip)}, \end{cases} \tag{7}$$

$\mu_s > \mu_k$ are static and kinetic coefficients; $v_0 \approx 0.1\,\text{m/s}$.

Because μ_{eff} *decreases* with $|v_{\text{rel}}|$ in the slip phase (negative velocity-dependent friction), energy is continuously pumped into the string — the mechanism behind self-sustained oscillations.

4. Stick-Slip Oscillations and the Helmholtz Motion

Stick phase: $v_{\text{rel}} = 0$; the string moves with the bow. Stick ends when the string's restoring force exceeds $F_N\mu_s$:

$$|F_{\text{string}}(x_b, t)| > F_N\mu_s. \tag{8}$$

Slip phase: $v_{\text{rel}} \neq 0$; the string evolves under (5) until $\dot{u}(x_b, t)$ catches v_b again, then re-sticks.

Under ideal conditions a sharp kink travels back and forth — **Helmholtz motion** (Helmholtz, 1877) — triggering slip once per period, producing a sawtooth waveform at f_1.

Key Result. The Helmholtz sawtooth Fourier series:

$$u_{\text{Helm}}(x_b, t) = \frac{2v_bL}{\pi^2 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(2\pi f_n t), \tag{9}$$

gives $A_n \propto 1/n$ — the **physical origin of the harmonic series** in the cello sound.

In our synthetic model (Part 2) we generalise to $A_n \propto n^{-\gamma}$ with instrument-specific γ.

5. The Schelleng Diagram: Stability Conditions

5.1. Two Failure Modes

1. **Too much force** — string sticks at wrong time: raucous sound (our "harsh" / "near-breaking" regime).
2. **Too little force** — string never sticks: surface sound.

5.2. Analytical Bounds

Let $\beta_b = x_b/L$ (bow position, typically 0.05–0.12).

Schelleng Bounds. Stable Helmholtz motion requires:

$$F_N^{\min} < F_N < F_N^{\max}, \tag{10}$$

$$F_N^{\min} = \frac{2\pi Z_0 v_b}{\mu_s - \mu_k}\,\beta_b, \tag{11}$$

$$F_N^{\max} = \frac{Z_0 v_b}{2(\mu_s - \mu_k)} \cdot \frac{1}{\beta_b}, \tag{12}$$

where $Z_0 = \mu c$ is the string's **characteristic impedance**.

Key Result. The width of the stability window:

$$\frac{F_N^{\max}}{F_N^{\min}} = \frac{\pi}{4\beta_b^2}. \tag{13}$$

For $\beta_b = 0.07$: ratio ≈ 160. The transition between regimes is **continuous** (smooth friction curve), which explains why our k-means gives ARI = 0.534 rather than 1.0 — the physics does not admit a sharp boundary.

6. Inharmonicity: Real Strings Are Not Ideal

Finite bending stiffness B adds a fourth-order term to (2):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \frac{B}{\mu}\frac{\partial^4 u}{\partial x^4}. \tag{14}$$

Inharmonic Partial Frequencies

$$f_n = nf_1\sqrt{1 + \beta_{\text{inh}}\, n^2}, \qquad \beta_{\text{inh}} = \frac{\pi^2 B}{TL^2}. \tag{15}$$

For the cello C string: $\beta_{\text{inh}} \approx 3 \times 10^{-4}$, giving $\Delta f_{10} \approx 10\,\text{Hz}$ — measurable and used as feature η in $\Phi[x]$. Different string constructions yield different β_{inh}, making η an instrument-specific fingerprint independent of pitch.

7. Summary and Connection to Part 2

1. Wave equation and normal modes (eq. 2–3).
2. Driven string with nonlinear bow force (eq. 5–7).
3. Helmholtz motion: $A_n \propto 1/n$ (eq. 9).
4. Schelleng bounds: analytical stability window with continuous transitions (eq. 10–13).
5. Inharmonicity from string stiffness (eq. 15).

Connection to Parts 2–3. Each result above directly motivates a computational choice: $A_n \propto n^{-\gamma}$ generalises eq. (9); β_{inh} enters the synthetic frequency grid; the smooth Schelleng transition explains ARI = 0.534; the single dominant control parameter F_N/F_N^{min} explains why PC1+PC2 captures 98.4% of the inter-instrument variance.

References

Helmholtz, H. von (1877). *On the Sensations of Tone.* Dover reprint, 1954.

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Woodhouse, J. (1993). On the playability of violins, Part I. *Acustica*, **78**, 125–136.

Next: Part 2 — From Signal to Feature Vector